stichting
mathematisch
centrum

AFDELING MATHEMATISCHE BESLISKUNDE BW 5/70 SEPTEMBER

H.C. TIJMS

A LIMIT THEOREM FOR THE PERIODIC REVIEW INVENTORY MODEL WITH NO SET-UP COST

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

A LIMIT THEOREM FOR THE PERIODIC REVIEW INVENTORY MODEL WITH NO SET-UP COST *)

H.C. Tijms

Mathematical Centre, Amsterdam, The Netherlands.

The periodic review, single item, stationary inventory model is considered. There is a constant lead time, a linear purchase cost, no set-up cost, a holding and shortage cost function, no discounting of cost, and total backlogging of unfilled demand. In the finite period model there is included a salvage value for stock remaining or required at the end of the final period. A weak condition is imposed on the one period expected holding and shortage cost. In this paper a limit theorem is proved which relates the minimal total expected cost in the n-period model for large n to the minimal average expected cost per period in the infinite period model.

We consider a stationary inventory model in which the demands D_1, D_2,... for a single item in periods 1,2,... are independent, nonnegative, discrete random variables with the common probability distribution $\phi(j) = P\{D_t = j\}$, $(j \geq 0;\ t \geq 1)$. It is assumed that $\mu = ED_t$ is finite and positive. At the beginning of each period the stock on hand and on order is reviewed. An order may then be placed for any nonnegative integral quantity of stock. An order placed at the beginning of period t is delivered at the beginning of period $t + \lambda$, where λ is a known nonnegative integer. We assume that all excess demands are backlogged and satisfied by future deliveries.

Let x_t denote the stock on hand and on order prior to placing any order in period t. Let y_t denote the stock on hand and on order after ordering in period t. The range of both x_t and y_t is given by the set I of all integers.

The ordering decision in period t is based upon $H_t = (x_1,\ldots,x_t,\ y_1,\ldots,y_{t-1})$. The vector H_t represents the history of the process up to the beginning of period t. An ordering policy is a sequence $R = (R_1,\ R_2,\ldots)$ of finite integral valued functions to be used as follows. At the beginning of period t, after having observed the past history H_t, a quantity $R_t(H_t) - x_t \geq 0$ is ordered.

The following costs are considered. The cost of ordering z units is cz. Let g(i) be the holding and shortage cost in a period when i is the amount of stock on hand just after any additions to stock in that period.

Let $\phi^n(j) = P\{D_1+\ldots+D_n=j\}$, $(j \geq 0;\ n \geq 1)$, let $\phi^0(0) = 1$, and let $\phi^0(j) = 0$ for $j > 0$. We assume that

$$L(k) = \sum_{j=0}^{\infty} g(k-j)\ \phi^{\lambda}(j)$$

---

*) Report BW 5/70 of the Operational Research Department of the Mathematical Centre.

exists and is finite for each $k \in I$. $L(k)$ can be interpreted as the expected holding and shortage cost in period $t + \lambda$ when k is the stock on hand and on order after ordering in period t. We assume that there exists a finite integer $\bar{x}$ such that $L(k) \geq L(\bar{x})$ for $k < \bar{x}$ and $L(k)$ is nondecreasing in k for $k \geq \bar{x}$.

In the n-period model there are made ordering decisions only in the periods $1,\dots,n$ and the problem is to find a policy which minimizes the total expected cost over periods $\lambda+1,\dots,\lambda+n$. In the formulation of the n-period model we follow VEINOTT[5,6] by assuming that stock left over at the end of period $\lambda + n$ can be salvaged with a return of the initial purchase cost. Similarly, any backlogged demand remaining at the end of period $\lambda + n$ can be satisfied by a purchase at this same cost.

Denote by $f_n(i|R)$ the total expected cost over periods $\lambda+1,\dots,\lambda+n$ when i is the amount of stock on hand and on order prior to ordering in period 1 and the policy R is followed. We have [5,6]

$$f_n(i|R) = \sum_{t=1}^{n} E_R[(y_t-x_t)c + L(y_t)|x_1 = i] - c[E_R(x_{n+1}|x_1 = i) - \lambda\mu] =$$

$$= \sum_{t=1}^{n} E_R(L(y_t)|x_1 = i) - ci + nc\mu + c\lambda\mu, \tag{1}$$

where E_R denotes the expectation under policy R.

In the n-period model a policy R^* is called optimal if $f_n(i|R^*) = \min_R f_n(i|R)$ for all $i \in I$.

In the infinite period model a policy R^* is called optimal if $a(i|R^*) = \min_R a(i|R)$ for all $i \in I$, where $a(i|R)$ is defined by

$$a(i|R) = \liminf_{n\to\infty} (1/n) f_n(i|R) . \tag{2}$$

When the limit in (2) exists $a(i|R)$ represents the average expected cost per period when i is the stock on hand and on order prior to ordering in period 1 and the policy R is followed.

The (x,x) policy, called the single critical number policy, has the following form. When at the beginning of a period the stock on hand and on order $i < x$, order then $x-i$ units; otherwise, do not order. The same parameter x is used in each period.

Let $\bar{x}$ be any integer such that $L(k)$ is minimal at $k = \bar{x}$ and $L(k)$ is nondecreasing in k for $k > \bar{x}$. It is known that the $(\bar{x},\bar{x})$ policy is optimal for the n-period and the infinite period models [1 pp. 387-390, 5] .

Let

$$f_n(i) = \min_R f_n(i|R), \quad i \in I. \tag{3}$$

It is not difficult to verify that $E_{(\bar{x},\bar{x})} (L(y_t)|x_1 = i)$ converges to $L(\bar{x})$ as $t \to \infty$ for each $i \in I$ (see the proof of the theorem in the next section). From (1) and (2) it follows now that $a(i|\bar{x},\bar{x}) = L(\bar{x}) + c\mu$, $i \in I$. Hence

$$\min_R a(i|R) = L(\bar{x}) + c\mu \qquad \text{for all } i \in I. \tag{4}$$

In the next section we shall prove that $[f_n(i) - n(L(\bar{x}) + c\mu)]$ converges to a specified function $v(i)$, $i \in I$. IGLEHART [3, pp. 16-26] has proved in a quite different way an analogous result for the case in which $L(k)$ is convex and in the n-period model stock left over at the end of period $\lambda + n$ has no value and backlogged demand remaining at the end of period $\lambda + n$ is never satisfied. We note that in the finite period model without salvage value the optimal single critical numbers are usually not the same for any period.

A LIMIT THEOREM

Let us define the renewal quanties $m(j)$ and $M(j)$ by

$$m(j) = \sum_{n=1}^{\infty} \phi^n(j) \text{ and } M(j) = \sum_{k=0}^{j} m(k), \quad j \geq 0 \quad . \tag{5}$$

THEOREM. Let $\bar{x}$ be any integer such that $L(k)$ is minimal at $k = \bar{x}$ and $L(k)$ is non-decreasing in k for $k > \bar{x}$. Then

$$\lim_{n\to\infty} [f_n(i) - n(L(\bar{x}) + c\mu)] = v(i), \qquad i \in I, \tag{6}$$

where

$$v(i) = \begin{cases} -ci + c\lambda\mu, & i < \bar{x}, \\ L(i) + \sum_{j=0}^{i-\bar{x}} L(i-j)\, m(j) - L(\bar{x})\, \{1 + M(i-\bar{x})\} - ci + c\lambda\mu, & i \geq \bar{x} . \end{cases} \tag{7}$$

PROOF. Let $p_{i\bar{x}}^{(t)}$ denote the probability that $y_t = \bar{x}$, given that $x_1 = i$ and that the policy $(\bar{x},\bar{x})$ is followed. If $i \leq \bar{x}$, then $p_{i\bar{x}}^{(t)} = 1$ for all $t \geq 1$. If $i > \bar{x}$, then $p_{i\bar{x}}^{(t)} = 1 - \Phi^{t-1}(i-\bar{x}+1)$, $t \geq 2$, where $\Phi^n(j) = P\{D_1+\ldots+D_n \leq j\}$, $(j \geq 0,\ n \geq 1)$. For every $j \geq 0$ there exists an integer r such that $\Phi^r(j) < 1$, since $\phi(0) < 1$. Further, we have that $\Phi^n(j) \leq \Phi^{n-1}(j)$, $(j \geq 0;\ n \geq 2)$. It will now be clear that $p_{i\bar{x}}^{(t)}$ converges exponentially fast to 1 as $t \to \infty$ for each $i \in I$.

Let for any α, $0 < \alpha < 1$, the function $G_\alpha(k)$ be defined by

$$G_\alpha(k) = L(k) + (1-\alpha)ck, \qquad k \in I. \tag{8}$$

Define for any α, $0 < \alpha < 1$ the function $V_\alpha(i)$ by

$$V_\alpha(i) = \sum_{t=1}^{\infty} \alpha^{t-1}\, E_{(\bar{x},\bar{x})}\, [L(y_t) + (y_t - x_t)c \mid x_1 = i], \quad i \in I. \tag{9}$$

We note that $V_\alpha(i)$ exists and is finite, since $\alpha < 1$ and under the condition $x_1 = i$ we have that $\bar{x} \leq y_t \leq \max(i,\bar{x})$, $t \geq 1$. The quantity $V_\alpha(i)$ can be interpreted as the total expected discounted cost over periods $\lambda+1, \lambda+2, \ldots$, all discounted to the beginning of period $\lambda+1$, when i is the stock on hand and on order prior to ordening in period 1 and the policy $(\bar{x},\bar{x})$ is followed. Using the fact that $x_{t+1} = y_t - D_t$, $t \geq 1$, we have [5,6]

$$V_\alpha(i) = \sum_{t=1}^{\infty} \alpha^{t-1}\, E_{(\bar{x},\bar{x})}\, (G_\alpha(y_t) \mid x_1 = i) - ci + \alpha c\mu/(1-\alpha), \quad i \in I. \tag{10}$$

It is shown in [6] that

$$V_\alpha(i) = \begin{cases} -ci + \alpha c\mu/(1-\alpha) + G_\alpha(\bar{x})/(1-\alpha), & i < \bar{x}, \\ -ci + \alpha c\mu/(1-\alpha) + G_\alpha(i) + \sum_{j=0}^{i-\bar{x}} G_\alpha(i-j)\, m_\alpha(j) + & \\ \quad + [G_\alpha(\bar{x})/(1-\alpha)]\,[\alpha - (1-\alpha)\, M_\alpha(i-\bar{x})]\ , & i \geq \bar{x}\ , \end{cases} \tag{11}$$

where

$$m_\alpha(j) = \sum_{n=1}^{\infty} \alpha^n \phi^n(j) \text{ and } M_\alpha(j) = \sum_{k=0}^{j} m_\alpha(k)\ ,\quad j \geq 0\ . \tag{12}$$

It follows from (5), (7), (8), (11) and (12) that

$$\lim_{\alpha \uparrow 1} \{V_\alpha(i) - (G_\alpha(\bar{x}) + \alpha c\mu)/(1-\alpha)\} = v(i) - c\lambda\mu,\quad i \in I. \tag{13}$$

Put for abbreviation $E_{(\bar{x},\bar{x})}\ (L(y_t)|x_1 = i) = L_t(i)$. By (8) and (10) we have that

$$V_\alpha(i) - (G_\alpha(\bar{x}) + \alpha c\mu)/(1-\alpha) = \sum_{t=1}^{\infty} \alpha^{t-1}\ \{L_t(i) - L(\bar{x})\} - c\bar{x} - ci + \\ + c(1-\alpha) \sum_{t=1}^{\infty} \alpha^{t-1}\ E_{(\bar{x},\bar{x})}\ (y_t|x_1 = i),\ i \in I. \tag{14}$$

Since $E_{(\bar{x},\bar{x})}(y_t|x_1 = i)$ converges to $\bar{x}$ as $n \to \infty$ for each $i \in I$, we have by a well-known Abel theorem that the last term in the right side of (14) converges to $c\bar{x}$ as $\alpha \uparrow 1$. So we have by (13) and (14) that

$$\lim_{\alpha \uparrow 1} \sum_{t=1}^{\infty} \alpha^{t-1}\{L_t(i) - L(\bar{x})\} = v(i) - c\lambda\mu + ci,\quad i \in I. \tag{15}$$

Since $L_t(i) - L(\bar{x})$ converges exponentially fast to zero as $t \to \infty$ for each $i \in I$, we can apply a Tauber theorem [4, p.10]. This results in

$$\sum_{t=1}^{\infty} \{L_t(i) - L(\bar{x})\} = v(i) - c\lambda\mu + ci,\qquad i \in I. \tag{16}$$

By $\sum_{t=1}^{n} L_t(i) + nc\mu - ci + c\lambda\mu = f_n(i|\bar{x},\bar{x}) = f_n(i)$ and (16) the theorem is proved.

We note that a review of the proof reveals that for any (x,x) policy we have that $[f_n(i|x,x) - n(L(x) + c\mu)]$ converges to $v(i)$ as $n \to \infty$, where $v(i)$ is given by (7) provided that we replace $\bar{x}$ by x in (7). The numbers $v(i)$ play the same role as the so-called "relative values" in HOWARD's model [2]. Finally, we note that a discrete demand distribution is not necessary in the proof. The proof may be adapted to any general demand distribution.

REFERENCES.

1. F.S. HILLIER and G.J. LIEBERMAN, Introduction to Operations Research, Holden-Day, San Francisco, 1967.

2. R.A. HOWARD, Dynamic Programming and Markov Processes, Technology Press, Cambridge, Mass., 1960.

3. D.L. IGLEHART, "Dynamic Programming and Stationary Analysis of Inventory Problems", Chap. 1 in H. SCARF, D. GILFORD and M. SHELLY (eds.), Multistage Inventory Models and Techniques, Stanford Univ. Press, Stanford, Calif. 1963.

4. E.C. TITCHMARSCH, The Theory of Functions, Oxford, 2nd edition, 1939.

5. A.F. VEINOTT, Jr., "The Optimal Inventory Policy for Batch Ordering", Opns. Res. 13(1965), 424-432.

6. A.F. VEINOTT, Jr. and H.M. WAGNER, "Computing Optimal (s,S) Inventory Policies", Management Sci. 11(1965), 525-552.